Supplementary Material
for
Optimal Taylor Rules when Targets are Uncertain

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August 2019

Proofs of the Propositions

Lemma 1 *Under the assumptions in section 2, the unique competitive equilibrium of the model is characterized by equations (9) and (10), where* $\Phi_j = \sigma\left(1-\varrho_j\right)\left(1-\beta\varrho_j\right) - \kappa\varrho_j$ *for* $j = \{r, u, my, m\pi\}$ *are constants that are independent of monetary policy.*

Proof: Consider the system (1), (2) and (TR1) with the exogenous processes (3) to (6). Using (TR1) to eliminate i_t in (2) gives the system

$$\pi_t = \kappa y_t + \beta E_t\left[\pi_{t+1}\right] + u_t, \tag{A1}$$

$$y_t = E_t\left[y_{t+1}\right] - \frac{1}{\sigma}\left(\phi_\pi\left(\pi_t + m_t^\pi\right) + \phi_y\left(y_t + m_t^y\right) - r_t^e - E_t\left[\pi_{t+1}\right]\right). \tag{A2}$$

We conjecture an equilibrium solution

$$\pi_t = s_{\pi r} r_t^e + s_{\pi m^\pi} m_t^\pi + s_{\pi m^y} m_t^y + s_{\pi u} u_t$$

$$y_t = s_{yr} r_t^e + s_{ym^\pi} m_t^\pi + s_{ym^y} m_t^y + s_{yu} u_t$$

and solve for the eight unknown coefficients $s_{h,j}$ for $h = \pi, y$ and $j = r, u, my, m\pi$. Substituting the conjectured relationships into (A1) and (A2) and using (3) to (6) to evaluate the expectations gives

$$\begin{aligned} 0 \;=\; & \left(s_{\pi r} - \kappa s_{yr} - \beta s_{\pi r}\varrho_r\right) r_t^e + \left(s_{\pi m^\pi} - \kappa s_{ym^\pi} - \beta s_{\pi m^\pi}\varrho_{m^\pi}\right) m_t^\pi \\ & + \left(s_{\pi m^y} - \kappa s_{ym^y} - \beta s_{\pi m^y}\varrho_{m^y}\right) m_t^y + \left(s_{\pi u} - \kappa s_{yu} - 1 - \beta s_{\pi u}\varrho_u\right) u_t \end{aligned}$$

and

$$\begin{aligned} 0 \;=\; & \left[\left(\varrho_r - 1\right)\sigma s_{yr} - \phi_\pi s_{\pi r} + 1 - \phi_y s_{yr} + s_{\pi r}\varrho_r\right] r_t^e + \left[\left(\varrho_{m^\pi} - 1\right)\sigma s_{ym^\pi} - \phi_\pi s_{\pi m^\pi} - \phi_\pi - \phi_y s_{ym^\pi} + s_{\pi m^\pi}\varrho_{m^\pi}\right] m_t^\pi \\ & + \left[\left(\varrho_{m^y} - 1\right)\sigma s_{ym^y} - \phi_\pi s_{\pi m^y} - \phi_y - \phi_y s_{ym^y} + s_{\pi m^y}\varrho_{m^y}\right] m_t^y + \left[\left(\varrho_u - 1\right)\sigma s_{yu} - \phi_\pi s_{\pi u} - \phi_y s_{yu} + s_{\pi u}\varrho_u\right] u_t. \end{aligned}$$

Each coefficient in these expressions must be zero. This gives a system of eight equations in eight unknowns. Solving for these unknowns gives the coefficients in (9) and (10). (By collecting terms for each of the shocks r_t^e, m_t^π, m_t^y, and u_t one can split the system into four sub-systems, each with two equations and two unknowns. The four subsystems can then be solved separately.)

Uniqueness follows from the determinacy condition (7). ■

Proposition 1
(i) If the only shocks to the model are cost-push shocks, then the optimal policy requires that the Taylor rule coefficients lie on the affine manifold (11).
(ii) For any ϕ_π^* *and* ϕ_y^* *satisfying (11), the equilibrium is*

$$\pi_t = \frac{1-\beta\varrho_u}{\kappa^2 + \alpha\left(1-\beta\varrho_u\right)^2} u_t, \qquad y_t = -\frac{\alpha\kappa}{\alpha\kappa^2 + \left(1-\beta\varrho_u\right)^2} u_t. \tag{A3}$$

Proof: Substituting (9) and (10) (and using $r_t^e = m_t^y = m_t^\pi = 0$) into (8) gives

$$V\left[u_t\right]\left(\alpha\left(\frac{\phi_y + \left(1-\varrho_u\right)\sigma}{\Phi_u + \phi_y\left(1-\beta\varrho_u\right) + \kappa\phi_\pi}\right)^2 + \left(\frac{\varrho_u - \phi_\pi}{\Phi_u + \phi_y\left(1-\beta\varrho_u\right) + \kappa\phi_\pi}\right)^2\right)$$

The first order condition for ϕ_π requires

$$0 = \alpha\kappa\left(\phi_y + \left(1-\varrho_u\right)\sigma\right) + \left(\varrho_u - \phi_\pi\right)\left(1-\beta\varrho_u\right)$$

which implies (11). The first order condition for ϕ_y requires

$$0 = \alpha\kappa\phi_y (\phi_\pi - \varrho_u) + \alpha\kappa\sigma (1 - \varrho_u) (\phi_\pi - \varrho_u) - (\varrho_u - \phi_\pi)^2 (1 - \beta\varrho_u) .$$

This condition can be satisfied either by setting $\phi_\pi = \varrho_u$ or by (11). This establishes (i).

To establish (ii), use $\Phi_u = (1 - \varrho_u) (1 - \beta\varrho_u) \sigma - \kappa\varrho_u$ to write the equilibrium conditions as

$$\pi_t = \frac{\phi_y + (1 - \varrho_u) \sigma}{(1 - \varrho_u) (1 - \beta\varrho_u) \sigma + \kappa (\phi_\pi - \varrho_u) + \phi_y (1 - \beta\varrho_u)} u_t,$$

$$y_t = \frac{\varrho_u - \phi_\pi}{(1 - \varrho_u) (1 - \beta\varrho_u) \sigma + \kappa (\phi_\pi - \varrho_u) + \phi_y (1 - \beta\varrho_u)} u_t.$$

Substituting (11) gives (A3). This establishes (ii). ■

Proposition 2 *Suppose r_t^e and u_t are i.i.d. over time and have covariance $Cov\left[r_t^e, u_t\right]$. The Taylor rule coefficients satisfying (11') and $\phi_y \to \infty$ are optimal.*

Proof: When r_t^e and u_t are i.i.d. equations (9) and (10) simplify to

$$\pi_t = \frac{\kappa}{\sigma + \phi_y + \kappa\phi_\pi} r_t^e + \frac{\phi_y + \sigma}{\sigma + \phi_y + \kappa\phi_\pi} u_t$$

$$y_t = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi} r_t^e - \frac{\phi_\pi}{\sigma + \phi_y + \kappa\phi_\pi} u_t$$

Plugging them into objective (8), and simplifying, shows that the optimal ϕ_y and ϕ_π minimize

$$\left(\sigma + \phi_y + \kappa\phi_\pi\right)^{-2} \left[\left(1 + \alpha\kappa^2\right) V\left[r^e\right] + \left(\alpha\left(\phi_y + \sigma\right)^2 + \left(\phi_\pi\right)^2\right) V[u] + 2\left(\alpha\kappa\left(\phi_y + \sigma\right) - \phi_\pi\right) Cov\left[r_t^e, u_t\right]\right]$$

First we fix the overall strength of the policy response by setting $\sigma + \phi_y + \kappa\phi_\pi = A$ and minimizing

$$A^{-2} \left\{\left(1 + \alpha\kappa^2\right) V\left[r^e\right] + \left(\alpha\left(\phi_y + \sigma\right)^2 + \left(\phi_\pi\right)^2\right) V[u] + 2\left(\alpha\kappa\left(\phi_y + \sigma\right) - \phi_\pi\right) Cov\left[r_t^e, u_t\right]\right\}$$

subject to $\sigma + \phi_y + \kappa\phi_\pi = A$.

The Lagrangian is

$$\mathcal{L} = A^{-2} \left\{\left(1 + \alpha\kappa^2\right) V\left[r^e\right] + \left(\alpha\left(\phi_y + \sigma\right)^2 + \left(\phi_\pi\right)^2\right) V[u] + 2\left(\alpha\kappa\left(\phi_y + \sigma\right) - \phi_\pi\right) Cov\left[r_t^e, u_t\right]\right\} + \lambda\left[A - \sigma - \phi_y - \kappa\phi_\pi\right]$$

and the first order conditions w.r.t. ϕ_y and ϕ_π, respectively, are

$$A^{-2} \left\{2\alpha\left(\phi_y + \sigma\right) V[u] + 2\alpha\kappa Cov\left[r_t^e, u_t\right]\right\} = \lambda$$

$$A^{-2} \left\{2\left(\phi_\pi\right) V[u] - 2 Cov\left[r_t^e, u_t\right]\right\} = \kappa\lambda$$

Combining them yields (11').

Since the objective is decreasing in A, it is optimal to let ϕ_y approach infinity. ■

Proposition 3 *Suppose all shocks are white noise, that is, $\varrho_r = \varrho_u = \varrho_{my} = \varrho_{m\pi} = 0$. Then the minimization of (8) subject to (9) and (10) yields the optimal Taylor rule coefficients given by (12) and (13).*

Proof: Setting $\varrho_r = \varrho_u = \varrho_{my} = \varrho_{m\pi} = 0$ in (9) and (10) and substituting into the objective (8) and using the fact that the shocks are (by assumption) uncorrelated implies that the central bank wants to choose parameters ϕ_y and ϕ_π to minimize

$$\left(\sigma + \phi_y + \kappa\phi_\pi\right)^{-2} \left(\left[\alpha\kappa^2 + 1\right] V\left[r_t^n\right] + \left[\alpha\kappa^2 + 1\right] \left(\phi_\pi\right)^2 V\left[m_t^\pi\right] + \left[\alpha\kappa^2 + 1\right] \left(\phi_y\right)^2 V\left[m_t^y\right] + \left[\alpha\left(\phi_y + \sigma\right)^2 + \left(\phi_\pi\right)^2\right] V\left[u_t\right]\right)$$

The first order condition for ϕ_y is

$$\begin{aligned} & \left\{\left[\alpha\kappa^2 + 1\right] V\left[m_t^y\right] \phi_y + \alpha\left(\phi_y + \sigma\right) V\left[u_t\right]\right\} \left(\sigma + \phi_y + \kappa\phi_\pi\right) \qquad \text{(A4)} \\ = \; & \left[\alpha\kappa^2 + 1\right] V\left[r_t^n\right] + \left[\alpha\kappa^2 + 1\right] V\left[m_t^\pi\right] \left(\phi_\pi\right)^2 + \left[\alpha\kappa^2 + 1\right] V\left[m_t^y\right] \left(\phi_y\right)^2 + \left[\alpha\left(\phi_y + \sigma\right)^2 + \left(\phi_\pi\right)^2\right] V\left[u_t\right] \end{aligned}$$

The first order condition w.r.t. ϕ_π is

$$\begin{aligned}
& \frac{\phi_\pi}{\kappa}\left\{\left[\alpha\kappa^2+1\right]V\left[m_t^\pi\right]+V\left[u_t\right]\right\}\left(\sigma+\phi_y+\kappa\phi_\pi\right) \qquad \text{(A5)}\\
= & \left[\alpha\kappa^2+1\right]V\left[r_t^n\right]+\left[\alpha\kappa^2+1\right]V\left[m_t^\pi\right]\left(\phi_\pi\right)^2+\left[\alpha\kappa^2+1\right]V\left[m_t^y\right]\left(\phi_y\right)^2+\left[\alpha\left(\phi_y+\sigma\right)^2+\left(\phi_\pi\right)^2\right]V\left[u_t\right]
\end{aligned}$$

It is immediate to see that this implies

$$\phi_\pi=\kappa\frac{\left[\alpha\kappa^2+1\right]V\left[m_t^y\right]+\alpha V\left[u_t\right]}{\left[\alpha\kappa^2+1\right]V\left[m_t^\pi\right]+V\left[u_t\right]}\phi_y+\frac{\alpha\kappa\sigma V\left[u_t\right]}{\left[\alpha\kappa^2+1\right]V\left[m_t^\pi\right]+V\left[u_t\right]} \qquad \text{(A6)}$$

We can rewrite equation (A5) as

$$\begin{aligned}
& \kappa\left[\alpha\kappa^2+1\right]V\left[r_t^n\right]+\kappa\left[\alpha\kappa^2+1\right]V\left[m_t^y\right]\left(\phi_y\right)^2\\
= & \phi_\pi\left[\alpha\kappa^2+1\right]\left(\sigma+\phi_y+\kappa\phi_\pi\right)V\left[m_t^\pi\right]-\kappa\left[\alpha\kappa^2+1\right]V\left[m_t^\pi\right]\left(\phi_\pi\right)^2+\\
& \phi_\pi\left(\sigma+\phi_y+\kappa\phi_\pi\right)V\left[u_t\right]-\kappa\left[\alpha\left(\phi_y+\sigma\right)^2+\left(\phi_\pi\right)^2\right]V\left[u_t\right]
\end{aligned}$$

Cancelling like terms and simplifying gives

$$\begin{aligned}
& \frac{\kappa}{\sigma+\phi_y}\left[\alpha\kappa^2+1\right]V\left[r_t^n\right]+\frac{\kappa}{\sigma+\phi_y}\left[\alpha\kappa^2+1\right]V\left[m_t^y\right]\left(\phi_y\right)^2\\
= & \phi_\pi\left\{\left[\alpha\kappa^2+1\right]V\left[m_t^\pi\right]+V\left[u_t\right]\right\}-\kappa\alpha\phi_y V\left[u_t\right]-\kappa\alpha\sigma V\left[u_t\right]
\end{aligned}$$

Using condition (A6) we have

$$\phi_\pi\left(\left[\alpha\kappa^2+1\right]V\left[m_t^\pi\right]+V\left[u_t\right]\right)-\alpha\kappa\sigma V\left[u_t\right]=\kappa\left(\left[\alpha\kappa^2+1\right]V\left[m_t^y\right]+\alpha V\left[u_t\right]\right)\phi_y$$

Eliminating this term gives

$$\begin{aligned}
& \frac{\kappa}{\sigma+\phi_y}\left[\alpha\kappa^2+1\right]V\left[r_t^n\right]+\frac{\kappa}{\sigma+\phi_y}\left[\alpha\kappa^2+1\right]V\left[m_t^y\right]\left(\phi_y\right)^2\\
= & \kappa\left(\left[\alpha\kappa^2+1\right]V\left[m_t^y\right]+\alpha V\left[u_t\right]\right)\phi_y-\kappa\alpha\phi_y V\left[u_t\right].
\end{aligned}$$

Finally, we cancel terms to get (12). To find (13) use condition (A6) and rearrange terms. ■

Model with signal extraction To solve the model with signal extraction, we closely follow the setup in Svensson and Woodford (2003, 2004). The proofs of the results use the following notation and calculations. The model can be written as

$$\begin{pmatrix} X_{t+1} \\ \tilde{E}\mathbb{E}_t\left[x_{t+1}\right] \end{pmatrix}=A\begin{pmatrix} X_t \\ x_t \end{pmatrix}+B\left(i_t-\rho\right)+\begin{pmatrix} s_{t+1} \\ 0 \end{pmatrix} \qquad \text{(A7)}$$

where $X_t=\left(r_t^e,u_t,m_t^y,m_t^\pi\right)'$; $s_{t+1}=\left(\varepsilon_{t+1}^r,\varepsilon_{t+1}^u,\varepsilon_{t+1}^y,\varepsilon_{t+1}^\pi\right)'$; $x_t=\left(y_t,\pi_t\right)'$; and $\tilde{E}=\begin{pmatrix} 0 & 1 \\ \sigma & 1 \end{pmatrix}$. Partition the matrices A and B as follows

$$A=\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B=\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

and let

$$A_{11}=\begin{pmatrix} \varrho_r & 0 & 0 & 0 \\ 0 & \varrho_u & 0 & 0 \\ 0 & 0 & \varrho_{m^y} & 0 \\ 0 & 0 & 0 & \varrho_{m^\pi} \end{pmatrix}, A_{12}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B_1=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A_{21}=\begin{pmatrix} 0 & -\frac{1}{\beta} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, A_{22}=\begin{pmatrix} -\frac{\kappa}{\beta} & \frac{1}{\beta} \\ \sigma & 0 \end{pmatrix} \; B_2=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The flow objective is

$$y_t^2+\alpha\pi_t^2=x_t'Wx_t,\; W=\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \qquad \text{(A8)}$$

and the observation equations are

$$Z_t = \begin{pmatrix} y_t^m \\ \pi_t^m \end{pmatrix} = D \begin{pmatrix} X_t \\ x_t \end{pmatrix}, \text{ with } D = (D_1, D_2), \; D_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A9}$$

Notice that the measurement errors are included in X_t. This allows for arbitrary persistence of both types of measurement error.

The central bank's information set is $I_t^{CB} = \left\{\Theta, y_{t-j}^m, \pi_{t-j}^m : j \geq 0\right\}$ where Θ is a vector of all model parameters.

We write the Taylor rule as

$$i_t - \rho = F x_{t|t} \tag{A10}$$

where $F = \left(\psi_y, \psi_\pi\right)$. Notice the difference to Svensson and Woodford who write the policy as a linear function of the estimates of the state variables X_t.

Next, we conjecture that

$$x_{t|t} = G X_{t|t} \tag{A11}$$

Then the upper block of (A7) gives

$$X_{t+1} = A_{11} X_t + s_{t+1}$$

and taking expectations (based on I_t^{CB}) yields

$$X_{t+1|t} = A_{11} X_{t|t}.$$

Taking expectations of the lower block of (A7) gives

$$\tilde{E} x_{t+1|t} = A_{21} X_{t|t} + A_{22} x_{t|t} + B_2 \left(i_t - \rho\right).$$

Combining these equations with (A10) and (A11), we arrive at

$$x_{t|t} = \left[A_{22}\right]^{-1} \left(-A_{21} + \tilde{E} G A_{11} - B_2 F G\right) X_{t|t}.$$

Hence, G must satisfy

$$G = \left[A_{22}\right]^{-1} \left(-A_{21} + \tilde{E} G A_{11} - B_2 F G\right). \tag{A12}$$

Next we conjecture that

$$x_t = G^1 X_t + \left(G - G^1\right) X_{t|t} \tag{A13}$$

and rewrite the observation equation (A9) as

$$\begin{aligned} Z_t &= D_1 X_t + D_2 x_t \\ &= \left(D_1 + D_2 G^1\right) X_t + D_2 \left(G - G^1\right) X_{t|t} \end{aligned}$$

Following Svensson and Woodford,

$$Z_t = L X_t + M X_{t|t} \tag{A14}$$

where

$$L = \left(D_1 + D_2 G^1\right) \tag{A15}$$

and

$$M = D_2 \left(G - G^1\right).$$

The state equation of the Kalman filter is

$$X_{t+1} = A_{11} X_t + s_{t+1}$$

and the observation equation is (A14). Svensson and Woodford (2004) show that the Kalman filter updating equation takes the form

$$X_{t|t} = X_{t|t-1} + K L \left(X_t - X_{t|t-1}\right) \tag{A16}$$

(their equation 26 with $v_t = 0$) and that it is possible to write

$$X_{t+1|t+1} = (I + KM)^{-1} \left[(I - KL) A_{11} X_{t|t} + K Z_{t+1}\right] \tag{A17}$$

(their equation 30) where

$$K = P L' \left(L P L'\right)^{-1}. \tag{A18}$$

Furthermore,

$$P=\mathbb{E}\left[\left(X_t-X_{t|t-1}\right)\left(X_t-X_{t|t-1}\right)'\right]=A_{11}\left[P-PL'\left(LPL'\right)^{-1}LP\right]A_{11}'+\Sigma_s \tag{A19}$$

where Σ_s is the covariance matrix of the errors s_{t+1}.

Finally, one needs to find G^1. Again, following Svensson and Woodford (2003, 2004) we obtain

$$G^1=\left[A_{22}\right]^{-1}\left(-A_{21}+\tilde{E}GKLA_{11}+\tilde{E}G^1\left(I-KL\right)A_{11}\right). \tag{A20}$$

Lemma 2 *The central bank's estimates of inflation and the output gap satisfy (14) and (15) where Φ_r and Φ_u are defined as in Lemma 1.*

Proof: Take conditional expectations based on the central bank's information set I_t^{CB} of equations (1) to (4) to obtain

$$\pi_t=\kappa y_t+\beta\pi_{t+1|t}+u_{t|t}$$

$$y_t=y_{t+1|t}-\frac{1}{\sigma}\left(i_t-\rho-r^e_{t|t}-\pi_{t+1|t}\right)$$

$$r^e_{t+1|t}=\varrho_r r^e_{t|t}\text{ , and }u_{t+1|t}=\varrho_u u_{t|t}$$

Imposing the Taylor rule (TR2) and repeating steps in the proof of Lemma 1 yields the desired result. ■

Lemma A1 *The welfare objective (8) can be decomposed as follows*

$$(1-\beta)\,\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t\left(y_t^2+\alpha\pi_t^2\right)\right]=\left(\alpha\mathbb{E}\left[\pi^2_{t|t}\right]+\mathbb{E}\left[y^2_{t|t}\right]\right)+\left(\alpha\mathbb{E}\left[\left(\pi_t-\pi_{t|t}\right)^2\right]+\mathbb{E}\left[\left(y_t-y_{t|t}\right)^2\right]\right)$$

Proof: See Svensson and Woodford (2003, 2004). ■

Lemma A2 $\alpha\mathbb{E}\left[\left(\pi_t-\pi_{t|t}\right)^2\right]+\mathbb{E}\left[\left(y_t-y_{t|t}\right)^2\right]=tr\left[P\left(G^1\left(I-KL\right)\right)'WG^1\left(I-KL\right)\right]$ *where I is the identiy matrix.*

Proof: Define $T=\alpha\mathbb{E}\left[\left(\pi_t-\pi_{t|t}\right)^2\right]+\mathbb{E}\left[\left(y_t-y_{t|t}\right)^2\right]$ and notice that $T=\mathbb{E}\left[\left(x_t-x_{t|t}\right)'W\left(x_t-x_{t|t}\right)\right]$ where W is defined in (A8) above. Equation (A13) implies that

$$x_t-x_{t|t}=G^1X_t+\left(G-G^1\right)X_{t|t}-GX_{t|t}=G^1\left(X_t-X_{t|t}\right)$$

so we obtain

$$T=\mathbb{E}\left[\left(X_t-X_{t|t}\right)'\left(G^1\right)'WG^1\left(X_t-X_{t|t}\right)\right].$$

Next subtract $X_{t|t}$ from X_t and use (A16) to get

$$\begin{aligned}X_t-X_{t|t}&=X_t-\left(X_{t|t-1}+KL\left(X_t-X_{t|t-1}\right)\right)\\&=\left(I-KL\right)\left(X_t-X_{t|t-1}\right)\end{aligned}$$

Then

$$\begin{aligned}T&=\mathbb{E}\left[\left(X_t-X_{t|t-1}\right)'\left(I-KL\right)'\left(G^1\right)'WG^1\left(I-KL\right)\left(X_t-X_{t|t-1}\right)\right]\\&=\mathbb{E}\left[tr\left[\left(X_t-X_{t|t-1}\right)\left(X_t-X_{t|t-1}\right)'\left(I-KL\right)'\left(G^1\right)'WG^1\left(I-KL\right)\right]\right]\\&=tr\left[\mathbb{E}\left[\left(X_t-X_{t|t-1}\right)\left(X_t-X_{t|t-1}\right)'\right]\left(G^1\left(I-KL\right)\right)'WG^1\left(I-KL\right)\right]\end{aligned}$$

where $tr\left(\cdot\right)$ is the trace operator. Using (A19) yields the desired result. ■

Lemma A3 *Suppose all shocks are contemporaneously uncorrelated with each other and i.i.d over time. Then the term $\alpha\mathbb{E}\left[\left(\pi_t-\pi_{t|t}\right)^2\right]+\mathbb{E}\left[\left(y_t-y_{t|t}\right)^2\right]$ is independent of the Taylor rule coefficients.*

Proof: Given Lemma A2, it is sufficient to show that T is independent of the Taylor rule coefficients. If all shocks are i.i.d. over time then $A_{11}=0_{4\times 4}$ and G^1 as defined in (A20) reduces to

$$G^1=-\left[A_{22}\right]^{-1}A_{21}$$

which is independent of policy. Furthermore, $P = \Sigma_s$ (see A19) and because G^1 is independent of policy, so is L (defined in A15). Then $K = PL' \left(LPL'\right)^{-1}$is independent of policy (equation A18). As a result, T is independent of policy. ■

Lemma A4 *Suppose all shocks are contemporaneously uncorrelated with each other and i.i.d over time. Then $r^e_{t|t}$ and $u_{t|t}$ are independent of the Taylor rule coefficients.*
Proof: In the iid case with $A_{11} = 0_{4\times 4}$, equation (A17) reduces to

$$X_{t+1|t+1} = (I + KM)^{-1} KZ_{t+1}$$

Combining this with (A14) gives

$$X_{t|t} = (I + KM)^{-1} KZ_t = (I + KM)^{-1} K \left(LX_t + MX_{t|t}\right).$$

Rearranging yields

$$X_{t|t} = KLX_t.$$

In the proof of Lemma A3 we showed that K and L are independent of policy when shocks are i.i.d. Hence $r^e_{t|t}$ and $u_{t|t}$ are independent of the Taylor rule coefficients. ■

Proposition 4 *Suppose all shocks are contemporaneously uncorrelated with each other and i.i.d over time. Then the coefficients $\left\{\psi^*_y, \psi^*_\pi\right\}$ satisfying (11") and $\psi_y \to \infty$ are optimal.*
Proof: Lemmas A1 and A3 imply that minimizing objective (8) is equivalent to minimizing $\alpha\mathbb{E}\left[\pi^2_{t|t}\right] + \mathbb{E}\left[y^2_{t|t}\right]$. Furthermore, equations (14) and (15) simplify to

$$\pi_{t|t} = \frac{\kappa}{\sigma + \psi_y + \kappa\psi_\pi} r^e_{t|t} + \frac{\psi_y + \sigma}{\sigma + \psi_y + \kappa\psi_\pi} u_{t|t}$$

$$y_{t|t} = \frac{1}{\sigma + \psi_y + \kappa\psi_\pi} r^e_{t|t} - \frac{\psi_\pi}{\sigma + \psi_y + \kappa\psi_\pi} u_{t|t}$$

in the i.i.d. case. Lemma A4 shows that $r^e_{t|t}$ and $u_{t|t}$ are independent of the Taylor rule coefficients, but importantly, their covariance need not to be zero. The remainder of the proof is identical to the proof of Proposition 2. ■

Proposition 5 *Suppose the model is given by equations (1) to (6), (TR2), and the observation equations $\pi^m_t = \pi_t + m^\pi_t$ and $y^m_t = y_t + m^y_t$. The central bank uses the Kalman filter with information set I^{CB}_t to estimate the true state of the economy. Suppose further that $V[u_t] = 0$ and that $\varrho_{my} = \varrho_{m\pi} = 0$. Then, for any Taylor rule (TR2), with coefficients ψ_y and ψ_π, there exist coefficients $\left\{\tilde\phi_y, \tilde\phi_\pi, \tilde\nu\right\}$ such that the policy rule (TR3) generates the same equilibrium paths for all variables.*
Proof: Under the assumption that $V[u_t] = 0$ and $\varrho_{my} = \varrho_{m\pi} = 0$, it is possible to write the model as follows

$$X_t = r^e_t,\ s_{t+1} = \varepsilon^r_{t+1},\ x_t = \begin{pmatrix} y_t \\ \pi_t \end{pmatrix},\ \tilde E = \begin{pmatrix} 0 & 1 \\ \sigma & 1 \end{pmatrix},\ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$A_{11} = \varrho_r,\ A_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix}, B_1 = 0,\ A_{21} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ A_{22} = \begin{pmatrix} -\frac{\kappa}{\beta} & \frac{1}{\beta} \\ \sigma & 0 \end{pmatrix}\ B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The observation equations are

$$Z_t = \begin{pmatrix} y^m_t \\ \pi^m_t \end{pmatrix} + v_t = D\begin{bmatrix} X_t \\ x_t \end{bmatrix} + v_t,\ D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $v_t = (m^y_t, m^\pi_t)'$. Furthermore, $D_1 = (0,0)'$ and $D_2 = I_2$ Notice that A_{11} is a scalar. Because v_t is nonzero, the equations (and matrices) associated with the Kalman filter change somewhat (see Svensson and Woodford 2004) though (A17) continues to hold. Since A_{11} is a scalar, KL and KM are also scalars.

Using (A10) and (A11), write the interest rate as

$$i_t - \rho = \left(\psi_y, \psi_\pi\right)\left(y_{t|t}, \pi_{t|t}\right)' = Fx_{t|t} = FGX_{t|t}.$$

Using (A17) and substituting backwards yields

$$i_t = \sum_{j=0}^{\infty} FG\,(I+KM)^{-1}\left[(I-KL)\,A_{11}\,(I+KM)^{-1}\right]^{j} KZ_{t-j}$$

Since KL, and KM are scalars, set $\tilde{\nu} = (I-KL)\,A_{11}\,(I+KM)^{-1}$. Then write

$$i_t = \sum_{j=0}^{\infty} FG\,(I+KM)^{-1}\,\tilde{\nu}^{j} KZ_{t-j} = FG\,(I+KM)^{-1}\,KZ_t + \tilde{\nu} i_{t-1} = \tilde{\phi}_y y_t^m + \tilde{\phi}_\pi \pi_t^m + \tilde{\nu} i_{t-1}$$

where $\left(\tilde{\phi}_y, \tilde{\phi}_\pi\right) = FG\,(I+KM)^{-1}\,K$. ■